A model-based measure of network heterogeneity with an application to the Austrian interbank market

Riccardo Rastelli

joint work with Juraj Hledik

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Random graphs are typically used to visualize and model financial systems.

Figure: a network of interaction strength between major financial institutions worldwide. From: *science* 325.5939 (2009): 422-425.
A financial shock occurred at one particular node may spread to its neighbors causing a domino effect.

Recent research on systemic risk has focused much on the role played by the heterogeneity of the network.

- **Homogeneous networks:**
  - all banks behave similarly;
  - increased diversification;
  - shocks propagate easily;
  - shocks are generally absorbed without causing defaults.

- **Heterogeneous networks:**
  - more clustered and fragmented;
  - shocks do not propagate as easily;
  - more susceptible to targeted shocks.
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The data is provided by the central bank of Austria (OeNB).

It consists of quarterly summaries of the funds exchanged between 800 banks located in Austria, from 2008 to 2012 (16 time frames).

The data can be represented as a temporal network, where nodes correspond to banks, and edge weights to amounts borrowed/lent.

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Onset of the financial crisis

September 2008: network snapshot
The *true exposures* $x_{ij}^{(t)}$ are *not available* due to privacy reasons.

We focus our analysis on the following *relative exposures*:

$$y_{ij}^{(t)} = \frac{x_{ij}^{(t)}}{\sum_j x_{ij}^{(t)}}.$$  

Note that $y_{ij}^{(t)} \in [0, 1]$ and $\sum_j y_{ij}^{(t)} = 1$.

- These proportions can be extracted from the data *exactly*.
- By looking at the vector $y_i^{(t)} = \{y_{i1}^{(t)}, \ldots, y_{iN}^{(t)}\}$ we can observe how banks *diversify*. 
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• The most homogeneous network satisfies $y_{ij}^{(t)} = \frac{1}{N-1}$, $\forall i, j, t$.

• An entropy index can be used as a measure of homogeneity for the exposures of a bank:

$$E_i^{(t)} := -\sum_{j=1}^{N} y_{ij}^{(t)} \log \left( y_{ij}^{(t)} \right)$$

• $E_i^{(t)}$ is maximized if $y_{ij}^{(t)} = \frac{1}{N-1}$ for all $j$.

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An entropy-based measure of homogeneity

OeNB_800: average entropy

OeNB_100: average entropy

Time

Average entropy

2009 2010 2011 2012

0.40 0.45 0.50 0.55

OeNB_800: average entropy

Time

Average entropy

2009 2010 2011 2012

0.85 0.90 0.95 1.00

OeNB_100: average entropy
- The data is the adjacency cube $\mathcal{Y} = \{ y_{ij}^{(t)} \}_{i,j,t}$ with $t \in \{1, \ldots, T\}$ and $i, j \in \{1, \ldots, N\}$, $i \neq j$.

- $y_{ij}^{(t)}$ are assumed to be independent Dirichlet vectors:

$$\mathcal{L}_{\mathcal{Y}}(\alpha) = \prod_{t=1}^{T} \prod_{i=1}^{N} \left\{ \frac{\Gamma\left(\sum_{j} y_{ij}^{(t)}\right)}{\prod_{j} \Gamma\left(y_{ij}^{(t)}\right)} \prod_{j} \left[y_{ij}^{(t)}\right]^{\alpha_{ij}^{(t)} - 1} \right\}$$

with parameters satisfying:

$$\log\left(\alpha_{ij}^{(t)}\right) = \mu^{(t)} + \theta_i + \gamma_j$$
Interpretation of the likelihood parameters

Network model:

\[ y_{i}^{(t)} \sim \text{Dir} \left( \alpha_{i}^{(t)} \right); \quad \log \left( \alpha_{ij}^{(t)} \right) = \mu^{(t)} + \theta_i + \gamma_j \]

Consider a generic \( y \sim \text{Dir} \left( \alpha, \ldots, \alpha \right) \):

- Large \( \alpha \) \( \implies \) low variance \( \implies \) high homogeneity.
- Small \( \alpha \) \( \implies \) high variance \( \implies \) high heterogeneity.

\( \mu^{(t)} \) and \( \theta_i \) regulate the diversification of exposures of a node.

Consider an asymmetric \( y \sim \text{Dir} \left( \alpha_1, \ldots, \alpha_N \right) \)

- higher \( \alpha_j \) implies higher \( y_j \).

\( \gamma_j \) determines the attractiveness of nodes.
Interpretation of the likelihood parameters

Network model:

\[ y_{i,t} \sim \text{Dir} \left( \alpha_{i,t} \right); \quad \log \left( \alpha_{i,j,t} \right) = \mu(t) + \theta_i + \gamma_j \]

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Hierarchical structure

Network model:

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Random walk prior on the homogeneity level \( \mu \):

\[ \mu^{(1)} \sim \mathcal{N} \left( 0, 1000 \right); \quad \mu^{(t)} = \mu^{(t-1)} + \eta(t), \forall t > 1; \]

\[ \eta^{(t)} \sim \mathcal{N} \left( 0, 1/\tau_\eta \right); \quad \tau_\eta \sim \text{Gamma} \left( a_\eta, b_\eta \right); \]

IID Gaussian priors on \( \theta_i \) and \( \gamma_j \):

\[ \theta_i \sim \mathcal{N} \left( 0, 1/\tau_\theta \right); \quad \tau_\theta \sim \text{Gamma} \left( a_\theta, b_\theta \right); \]

\[ \gamma_j \sim \mathcal{N} \left( 0, 1/\tau_\gamma \right); \quad \tau_\gamma \sim \text{Gamma} \left( a_\gamma, b_\gamma \right). \]

Note: identifiability is ensured through \( \gamma_1 = -\sum_{j \neq 1} \gamma_j \).
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IID Gaussian priors on \( \theta_{i} \) and \( \gamma_{j} \):

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Note: identifiability is ensured through \( \gamma_{1} = -\sum_{j \neq 1} \gamma_{j} \).
Note: hyperparameters are all fixed to 0.01 to obtain noninformative priors.
We have used Markov chain Monte Carlo techniques to sample from $\pi(\mu, \theta, \gamma|\mathcal{Y})$.

We used a Metropolis-within-Gibbs framework, alternating the following steps:

- Update each $\mu^{(t)}$ using M-H and Gaussian proposal.
- Update each $\theta_i$ using M-H and Gaussian proposal.
- Update each $\gamma_j$ using M-H and Gaussian proposal.
- Sample $\tau_\eta$ from its conjugate full-conditional.
- Sample $\tau_\theta$ from its conjugate full-conditional.
- Sample $\tau_\gamma$ from its conjugate full-conditional.

From 400,000 iterations we extracted samples of 10,000 values for each model parameter.
Temporal evolution of diversification

OeNB_800: evolution of $\mu$

OeNB_100: evolution of $\mu$
Distribution of posterior averages of $\theta$

OeNB_800: theta posterior means

OeNB_100: theta posterior means
Results

Distribution of posterior averages of $\gamma$

OeNB_800: gamma posterior means

OeNB_100: gamma posterior means
Diversification vs relevance

OeNB_800: theta vs relevance

OeNB_100: theta vs relevance
Attractiveness vs relevance

OeNB_800: gamma vs relevance

OeNB_100: gamma vs relevance
Diversification vs attractiveness

OeNB_800: theta vs gamma

OeNB_100: theta vs gamma
Highlights

Advantages:

- New model for dynamic weighted networks.
- Parameters have interesting interpretations.
- Markov chains generally mix very well.

Limitations:

- Does not model the relevance of banks.
- Does not model sparseness directly.
- Trend may be not the same for all banks.

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